ST329: COMBINATORIAL STOCHASTIC PROCESSES
ST414 (TERM 2, 2013–2014)

PARTHA DEY
UNIVERSITY OF WARWICK

Content.

- Counting, Generating functions, Bell Polynomials and its applications, Moments and cumulants, Composite strutures, Gibbs partition.

Prerequisites.

- Basic knowledge of Probability theory, Expectation, Independence.
- The course ST111 (Probability A) covers all of them and is essential for this course.

Reading.

- This course is based on Chapter one of Jim Pitman's "Combinatorial Stochastic Process".
- Lecture notes online and example sheets at the end of the notes.

Contents

1. Counting

The main types of objects we count are sequences and sets. For example, there are n^k sequences of length k with entries from $[n] := \{1, 2, \ldots, n\}$. We use parentheses or round brackets $(\cdots)$ to denote that the entries of a sequence are *ordered* and we use curly braces $\{\cdots\}$ to denote that the elements of a set are *unordered*. This section is aimed at how to approach problems involving counting sequences and counting sets, by presenting two general principles and examples of how to apply them.

a) Multiplication principle

The first extremely useful step-by-step counting principle is for counting sequences:

The Multiplication Principle. The number of sequences $(x_1, x_2, \ldots, x_k)$ is the product of the number of choices of each x_i for $i \in [k]$.

For example, how do we see that there are $n!$ permutations of $[n]$? Well we do it step-by-step: there are n choices for the first entry of a permutation, $n-1$ choices for the next entry, and so on. By the multiplication principle, we just multiply these numbers together, so there are $n(n-1)(n-2)\cdots 2\cdot 1 = n!$ permutations.

Another example is to count the number of sequences $(x_1, x_2, \ldots, x_k)$ where $i \leqslant x_i \leqslant 2i$ for all $i \in [k]$. Clearly there are $i+1$ choices for x_i, so the multiplication principle gives us the answer, which is $(k+1)!$.

One more example is total number of subsets of $[n]$, which is 2^n. Here we used the multiplication principle and the fact that for any integer from $[n]$ there are two choices, the number can either be in the set or not. It is extremely important to realize that the multiplication principle does not work for counting sets, in general.

b) Unordering principle

Okay, but then how would we count sets? This leads into our second principle: if we count N sequences $(x_1, x_2, \ldots, x_k)$ and we know that none of the x_i's are the same, then the number of sets $\{x_1, x_2, \ldots, x_k\}$ is just $\frac{N}{k!}$.

The Unordering Principle. If there are N sequences $(x_1, x_2, \ldots, x_k)$ where none of the x_i's are equal, then there are $\frac{N}{k!}$ sets $\{x_1, x_2, \ldots, x_k\}$.

In other words, we divide by $k!$ to "get rid of the order". A good tip, therefore, when counting sets is to first count sequences and then divide by the appropriate factorial using the unordering principle. We stress that none of the x_i's can be equal when we apply the unordering principle. The fundamental example in applying the unordering principle is to count sets of size k in $[n]$ and the answer is $\binom{n}{k}$. Let's do it: first we count permutations $(x_1, x_2, \ldots, x_k)$ of length k with entries from $[n]$: by the multiplication principle there are $n(n-1)(n-2)\cdots(n-k+1)$ such permutations (check this). By the unordering principle, to get the number of sets of size k in $[n]$, we just divide the number of permutations of length k by $k!$ to get

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \binom{n}{k}.$$

In particular we have the following.

- The number of permutation of n distinguishable objects or the number of ways to arrange n distinguishable objects is $n!$ (by Multiplication principle).
- The number of permutation of length k from n distinguishable objects or the number of ways to arrange k objects out of n distinguishable objects is $n(n-1)(n-2)\cdots(n-k+1) = (n)_k$ (by Multiplication principle). The symbol $(n)_k$ is called falling factorial or Pochhammer symbol.
- The number of sets of size k in $[n]$ or number of ways to choose a collection of k objects from n distinguishable objects is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

1.1. **Counting with distinguishable objects.** Suppose that we have n distinguishable objects marked by numbers from $[n]$ and we want to fill k places with them. Each place can contain exactly one object. If no repetition is allowed for any object, when order matters the total number of configurations is $(n)_k$

and when the order doesn't matter the total number of configurations is $\frac{(n)_k}{k!} = \binom{n}{k}$, by the unordering principle.

If we allow repetition, the number of configurations in the above scenario when order matters is n^k (using the multiplication principle, each place can contain exactly one object in n ways). Let us now find the number of ways to choose k objects from the given n distinguishable objects when the order does not matter and repetition is allowed. We cannot use the *unordering principle* here as there can be repetition in the chosen k objects. What we need is how many times the first object appears, how many times the second object appears and so on to know the collection of k objects when the order does not matter and repetition is allowed.

Let $p_i \geqslant 0, i \geqslant 1$ be the number of times object i appear, so that we have $\sum_{i=1}^n p_i = k$. So we need to find number of ways to write k as sum of n non-negative numbers. We can do this by noticing that if we take k balls and $(n-1)$ bars $|$, and consider all possible permutation of the balls and bars. The $(n-1)$ bars ($|$'s) will divide the balls into k many piles, the first pile being all balls before the first $|$, the second pile being all balls between the first and second $|$, and so on, the last pile being all balls after the last $|$. The number of all such permutation is $\frac{(k+n-1)!}{k!(n-1)!} = \binom{n+k-1}{k}$. Thus we get the following table counting number of ways to fill k places with n distinguishable objects.

	No Repetition	Repetition allowed
Order matters	$(n)_k$	n^k
Order does not matter	$\binom{n}{k}$	$\binom{n+k-1}{k}$

TABLE 1. Filling k places with n distinguishable objects, each place contains exactly one object.

1.2. **Partition of a set.** A partition of $[n]$ is an unordered collection $\{A_1, A_2, \ldots, A_k\}$ of non-empty disjoint sets whose union is $[n]$ and such that no two of the sets A_i's intersect. The sets A_i are called the parts or blocks of the partition. We will use $\mathcal{P}_{[n]}^k$ to denote the set of partitions of the set $[n]$ into k blocks and $\mathcal{P}_{[n]} = \bigcup_{k=1}^n \mathcal{P}_{[n]}^k$ to denote the set of all partitions of $[n]$.

It is convenient to write 23 instead of the set $\{2,3\}$, otherwise the notation gets a bit ugly. So for example, $\{1,23\}$ is a partition of [3]. An ordered partition of $[n]$ is an ordering of the sets in a partition of $[n]$. For example, we count $(1,23)$ and $(23,1)$ as different ordered partitions of [3], even though $\{1,23\}$ is the same (unordered) partition as $\{23,1\}$. For us partition always means unordered partition unless explicitly mentioned.

How many ordered partitions of $[n]$ have their i-th part of size n_i for $i \in [k]$? The answer is the multinomial coefficient

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}.$$

This follows from the multiplication principle, since the product on the right represents the number of choices for the first part of the partition, then the second part, the third, and so on. But what about unordered partitions of $[n]$ into parts of sizes $n_1, n_2, \ldots, n_k$? This problem is not easy – in general we can't just divide the number of ordered partitions by $k!$. In the particular case $n_1 = n_1 = \cdots = n_k = n/k =: m$, we can apply the unordering principle to get the answer

$$\frac{n!}{k! \cdot m!^k}$$

since any ordering of the parts of the partition gives an ordered partition with all parts of size $m = n/k$. In the case where all the parts have different sizes – all n_i's are different – the number of unordered partitions is the same as the number of ordered partitions.

Let us consider the problem of filling k places with n distinguishable objects, but now each place can contain *more than one* object (with at least one object in each place) and no repetition is allowed. If we number the objects using $\{1,2,\ldots,n\} = [n]$, each place contains a set of objects and gives a partition of $[n]$ into k non-empty blocks. Thus if order matters the number of ways to fill is the number of ordered partition of $[n]$ into k non-empty blocks and if order doesn't matter the answer is number of partitions of $[n]$ into k non-empty blocks. We use $S(n,k)$ to denote number of (unordered) partitions of $[n]$ into k non-empty blocks. The numbers $S(n,k) = |\mathcal{P}_{[n]}^k|$ are known as Stirling numbers of second kind and will be discussed in detail in Section 2.1. The number $B(n) = |\mathcal{P}_{[n]}| = \sum_{k=1}^n S(n,k)$ is known as Bell number.

1.3. **Composition of an integer.** A composition of an integer n with k parts is a sequence $(x_1, x_2, \ldots, x_k)$ of positive integers such that $x_1 + x_2 + \cdots + x_k = n$. Let $\mathcal{C}_n^k$ denote the set of compositions of n into k parts and $\mathcal{C}_n = \bigcup_{k=1}^n \mathcal{C}_n^k$ denote the set of all compositions of n.

The x_i's are called the parts of the composition. To count these compositions, it seems at first we should just apply the multiplication principle: count the number of choices for x_1, and then the number of choices for x_2, and so on, and then multiply all these together. But there's a catch: once we've picked x_1, the number of choices of x_2 depends on what x_1 was picked! We have to find a different strategy. This is the third and perhaps trickiest principle: reduce the problem to a different counting problem to which we know the answer. We reduce this problem to counting sets of size k in $[n]$, to which the answer is $|\mathcal{C}_n^k| = \binom{n-1}{k-1}$. We do that by noticing that if we cut $[n]$ into k intervals, then the lengths of the intervals are exactly the x_i's. So how many ways can we cut $[n]$ into k intervals? Well we have to choose $k-1$ places to cut $[n]$ and there are $n-1$ possible places between numbers where cuts can be made. This validates our answer. We also mention that, when x_i's are allowed to be zero, instead of composition one will get a weak composition.

From here it follows that the number of ways to put n indistinguishable objects into k places where none of the places is empty and ordering matters is $\binom{n-1}{k-1}$. We haven't said anything about counting sets $\{x_1, x_2, , \ldots, x_k\}$ such that $x_1 + x_2 + \cdots + x_k = n$. This is with good reason: we cannot apply the unordering principle, so we get into the same problems as when we try to count unordered partitions. We discuss this in the next Section.

1.4. **Integer partition.** A partition of a positive integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. If order matters, the sum becomes a composition. Let $\mathcal{P}_n^k$ be the number of integer partitions of n into k parts. Thus

$$\mathcal{P}_n^k := \{(n_i)_{i=1}^k \mid n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1, \sum_{i=1}^k n_i = n\}.$$

The set $\mathcal{P}_n := \bigcup_{k=1}^n \mathcal{P}_n^k$ of all partitions of n can be written as

$$\mathcal{P}_n := \{(n_i)_{i=1}^\infty \mid n_1 \geqslant n_2 \geqslant \cdots \geqslant 0, \sum_{i=1}^\infty n_i = n\}.$$

Note that we can also write an integer partition of n as $(m_j)_{j=1}^n$ where m_j is the number of times j appears in the integer partition and $\sum_{j=1}^n j m_j = n$. Clearly $\sum_{j=1}^n m_j$ is the number of parts in the integer partition.

Let $P(n,k) := |\mathcal{P}_n^k|$ be the number of integer partitions of n into k parts. It is easy to see that the number of ways to put n indistinguishable objects into k places where none of the places is empty and ordering does not matter is $P(n,k)$. However, in this course we will not discuss much about $P(n,k)$ and its properties.

Combining we get the following table counting number of ways to fill k places with all of n objects where each place can contain more than one object and at least one object.

	distinguishable objects	indistinguishable objects
Order matters	$k!\,S(n,k)$	$\binom{n-1}{k-1}$
Order does'nt matter	$S(n,k)$	$P(n,k)$

TABLE 2. Filling k places with all of n objects, each place contains one or more object.

2. Exponential generating function

The **exponential generating function** associated with the weight sequence $\mathbf{w} = (w_n)_{n\geqslant 1}$ is defined as the formal power series

$$g_{\mathbf{w}}(x) = \sum_{n=0}^\infty w_n \frac{x^n}{n!}.$$

We also define, the **generating function** associated with the weight sequence $\mathbf{w} = (w_n)_{n\geqslant 1}$ as the formal power series

$$f_{\mathbf{w}}(x) = \sum_{n=0}^\infty w_n x^n.$$

Generally we will have $w_0 = 0$ and thus the sum will start from $n = 1$. Notation such as

$$c_n = [x^n]f(x)$$

should be read as "c_n is the coefficient of x^n in $f(x)$", meaning

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

where the power series might be convergent in some neighborhood of 0, or regarded as formally. Note that e.g.

$$\left[\frac{x^n}{n!}\right] f(x) = n![x^n]f(x).$$

2.1. Stirling numbers of second kind $S(n, k)$.

Definition 2.1. *Let*

$$S(n,k) = \textit{number of ways to partition } [n] \textit{ into } k \textit{ non-empty blocks.}$$

The numbers $S(n,k)$ is called **Stirling numbers of second kind** *and is also written as* $\left\{ {n \atop k} \right\}$.

It is easy to see that $S(n, 0) \equiv 0$ and $S(n, 1) \equiv 1$ for every $n \geqslant 1$, as there is exactly one partition with one block, namely the set itself.

Let us now try to find $S(n, 2)$. Note that $2!\, S(n, 2)$ is the number of ordered partition of $[n]$ into 2 non-empty blocks. In an ordered partition with 2 blocks, the first block can be any subset of $[n]$ except $\emptyset$ and $[n]$, and the second block is the complement of the first block. Number of subsets of $[n]$ is 2^n, by the multiplication principle (for each number in $[n]$ there are two options, either it is in the set or it is not). Thus

$$2!\, S(n, 2) = 2^n - 2 \text{ or } S(n, 2) = 2^{n-1} - 1 \text{ for } n \geqslant 1.$$

How to find $S(n, k)$ for general k? First we claim that the following relation is true.

Lemma 2.2. *For every $n \geqslant 1$ and $k \geqslant 1$ we have*

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k).$$

Proof. Recall that, $S(n, k)$ is the number of ways to partition $[n]$ into k non-empty blocks. Let us count the same number in a different way depending on whether $\{n\}$ is a block of the partition or not. If the set $\{n\}$ is a block of the partition, the number of choices for the remaining $(k-1)$ blocks are $S(n-1, k-1) =$ number of partitions of $[n-1]$ into $(k-1)$ non-empty blocks. If $\{n\}$ is NOT a block of the partition, we can choose a partition of $[n-1]$ into k non-empty blocks and put n in one of the k blocks to get a partition of n in $k \cdot S(n-1, k)$ ways. Thus we have

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k).$$ ■

We will use this relation to find an exact formula $S(n, k)$ in the next section. Let

$$g_k(x) := \sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \sum_{n=1}^{\infty} S(n,k)\frac{x^n}{n!}$$

be the exponential generating function associated with the Stirling numbers of second kind $S(n, k), n \geqslant 1$ for fixed $k \geqslant 1$. Using the relation $S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$ we have

$$\begin{aligned} g_k'(x) = \sum_{n=1}^{\infty} S(n,k)\frac{nx^{n-1}}{n!} &= \sum_{n=1}^{\infty} \Big(S(n-1,k-1) + k \cdot S(n-1,k)\Big)\frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=1}^{\infty} S(n-1,k-1)\frac{x^{n-1}}{(n-1)!} + k\sum_{n=1}^{\infty} S(n-1,k)\frac{x^{n-1}}{(n-1)!} \\ &= g_{k-1}(x) + kg_k(x). \end{aligned}$$

We claim the following.

Lemma 2.3. *We have*

$$g_k(x) = \frac{1}{k!}(e^x - 1)^k \textit{ for } k \geqslant 1.$$

In particular,

$$S(n,k) = \sum_{i=1}^{k} (-1)^{k-i} \frac{i^n}{i!(k-i)!} \textit{ for } n \geqslant 1, k \geqslant 1.$$

Proof. We have already proved that

$$g_k'(x) = g_{k-1}(x) + kg_k(x) \tag{1}$$

for all $k \geqslant 1$. We use induction to prove the result from here. For $k = 1$, it is easy to see that

$$g_1(x) = \sum_{n=1}^{\infty} 1 \cdot \frac{x^n}{n!} = e^x - 1.$$

Thus the lemma is true for $k = 1$. Assume that

$$g_{k-1}(x) = \frac{1}{(k-1)!}(e^x - 1)^{k-1}.$$

Using (1) and multiplying both sides by e^{-kx} we have

$$\begin{aligned} g_k'(x) - kg_k(x) &= \frac{1}{(k-1)!}(e^x - 1)^{k-1} \\ \text{or } (e^{-kx}g_k(x))' &= \frac{e^{-kx}}{(k-1)!}(e^x - 1)^{k-1} = \frac{e^{-x}}{(k-1)!}(1 - e^{-x})^{k-1}. \end{aligned}$$

Integrating both sides and noting that $g_k(0) = 0$ we have

$$e^{-kx}g_k(x) = \frac{1}{k!}(1 - e^{-x})^k \text{ or } g_k(x) = \frac{1}{k!}(e^x - 1)^k.$$

For the proof of the second part, note that

$$\begin{aligned} S(n,k) = \left[\frac{x^n}{n!}\right] g_k(x) &= \left[\frac{x^n}{n!}\right] \frac{1}{k!}(e^x - 1)^k \\ &= \left[\frac{x^n}{n!}\right] \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} e^{ix}(-1)^{k-i} \\ &= \left[\frac{x^n}{n!}\right] \sum_{i=0}^{k} (-1)^{k-i} \frac{e^{ix}}{i!(k-i)!} = \sum_{i=1}^{k} (-1)^{k-i} \frac{i^n}{i!(k-i)!}. \end{aligned}$$

■

3. Bell polynomials

Given the sequence $\mathbf{w}_\bullet = (w_1, w_2, \ldots)$ the (n,k)-th *partial Bell Polynomial* $B_{n,k}(\mathbf{w}_\bullet)$ is defined as

$$B_{n,k}(\mathbf{w}_\bullet) := \sum_{\{A_1, A_2, \ldots, A_k\} \in \mathcal{P}_{[n]}^k} \prod_{i=1}^{k} w_{|A_i|}$$

where $|A|$ is the number of elements in the set A and the sum is over $\mathcal{P}_{[n]}^k$ = all partitions of $[n]$ into k blocks. Note that $B_{n,k}(\mathbf{w}_\bullet)$ depends only on $w_1, w_2, \ldots, w_{n-k+1}$ as the largest block size can never be bigger than $n - k + 1$. It is easy to see that

$$B_{n,1}(\mathbf{w}_\bullet) = w_n \text{ for } n \geqslant 1$$

as $\mathcal{P}_{[n]}^1$ consists only of $\{[n]\}$. We also have (**exercise**)

$$B_{n,2}(\mathbf{w}_\bullet) = \begin{cases} \sum_{i=1}^{(n-1)/2} \binom{n}{i} w_i w_{n-i} & \text{if } n \text{ is odd} \\ \sum_{i=1}^{n/2} \binom{n}{i} w_i w_{n-i} - \frac{1}{2}\binom{n}{n/2} w_{n/2}^2 & \text{if } n \text{ is even.} \end{cases}$$

Note that for the sequence $\mathbf{1}_\bullet = (1, 1, \ldots)$, we have

$$B_{n,k}(\mathbf{1}_\bullet) = |\mathcal{P}_{[n]}^k| = S(n,k).$$

We can write $B_{n,k}(\mathbf{w}_\bullet)$ in an alternate form using compositions of n into k parts.

Lemma 3.1. *Let $\mathcal{C}_n^k$ denote the set of all compositions of n into k parts. We have*

$$B_{n,k}(\mathbf{w}_\bullet) = \frac{n!}{k!} \sum_{(n_1, n_2, \ldots, n_k) \in \mathcal{C}_n^k} \prod_{i=1}^{k} \frac{w_{n_i}}{n_i!}.$$

Proof. Note that we can write

$$B_{n,k}(\mathbf{w}_\bullet) := \frac{1}{k!} \sum_{(A_1, A_2, \ldots, A_k)} \prod_{i=1}^{k} w_{|A_i|}$$

where the sum is over all **ordered** partitions of $[n]$ into k blocks. Given an ordered partition $(A_1, A_2, \ldots, A_k)$ of $[n]$ into k blocks, the sequence $(|A_1|, |A_2|, \ldots, |A_k|)$ gives a composition of n into k parts. However, number of ordered partitions of $[n]$ giving the same composition $(n_1, n_2, \ldots, n_k)$ is (see Section 1.2)

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

Thus

$$B_{n,k}(\mathbf{w}_\bullet) := \frac{1}{k!} \sum_{(n_1, n_2, \ldots, n_k) \in \mathcal{C}_n^k} \frac{n!}{n_1! n_2! \cdots n_k!} \prod_{i=1}^{k} w_{n_i} = \frac{n!}{k!} \sum_{(n_1, n_2, \ldots, n_k) \in \mathcal{C}_n^k} \prod_{i=1}^{k} \frac{w_{n_i}}{n_i!}.$$

■

From Lemma 3.1 it follows that for the sequence $\bullet! = (1!, 2!, 3!, \ldots)$ we have

$$B_{n,k}(\bullet!) = \frac{n!}{k!} |\mathcal{C}_n^k| = \frac{n!}{k!} \binom{n-1}{k-1}.$$

This numbers $B_{n,k}(\bullet!)$ are also known as unsigned *Lah numbers* in the literature. From Lemma 3.1 we also obtain the following important result.

Lemma 3.2. *Let*

$$g(x) = \sum_{n=1}^{\infty} w_n \frac{x^n}{n!}$$

be the exponential generating function associated with the sequence $\mathbf{w}_\bullet = (w_1, w_2, \ldots)$. *Then we have*

$$\left[\frac{x^n}{n!}\right] \frac{g(x)^k}{k!} = B_{n,k}(\mathbf{w}_\bullet)$$

Proof. It is easy to see that

$$g(x)^k = g(x) \cdot g(x) \cdots g(x) = \sum_{n_1, n_2, \ldots, n_k \geqslant 1} w_{n_1} \frac{x^{n_1}}{n_1!} \cdot w_{n_2} \frac{x^{n_2}}{n_2!} \cdots w_{n_k} \frac{x^{n_k}}{n_k!}.$$

Thus

$$\left[\frac{x^n}{n!}\right] g(x)^k = n! [x^n] g(x)^k = n! \sum_{(n_1, n_2, \ldots, n_k) \in \mathcal{C}_n^k} \frac{w_{n_1}}{n_1!} \cdot \frac{w_{n_2}}{n_2!} \cdots \frac{w_{n_k}}{n_k!}$$

$$= n! \sum_{(n_1, n_2, \ldots, n_k) \in \mathcal{C}_n^k} \prod_{i=1}^{k} \frac{w_{n_i}}{n_i!}$$

Thus by Lemma 3.1 we have

$$\left[\frac{x^n}{n!}\right] \frac{g(x)^k}{k!} = \frac{n!}{k!} \sum_{(n_1, n_2, \ldots, n_k) \in \mathcal{C}_n^k} \prod_{i=1}^{k} \frac{w_{n_i}}{n_i!} = B_{n,k}(\mathbf{w}_\bullet).$$

■

Lemma 3.3. *Given a sequence of non-negative integers* $(m_j)_{j=1}^n$ *with* $\sum_{j=1}^n j m_j = n$ *and* $\sum_{j=1}^n m_j = k$ *the coefficient of* $w_1^{m_1} w_2^{m_2} \cdots w_n^{m_n}$ *in* $B_{n,k}(\mathbf{w}_\bullet)$ *is*

$$\left[w_1^{m_1} w_2^{m_2} \cdots w_n^{m_n}\right] B_{n,k}(\mathbf{w}_\bullet) = \frac{n!}{\prod_{j \geqslant 1} (j!)^{m_j} \cdot m_j!}.$$

Proof. Given a composition $(n_1, n_2, \ldots, n_k)$ of n into k parts, we define its skeleton as k_j = number of n_i's that are equal to j, for $j = 1, 2, \ldots, n$. Note that the resulting skeleton sequence $(k_j)_{j=1}^n$ satisfies $\sum_{j=1}^n j k_j = n$, $\sum_{j=1}^n k_j = k$.

Given a sequence of non-negative integers $(m_j)_{j=1}^n$ with $\sum_{j=1}^n jm_j = n$ and $\sum_{j=1}^n m_j = k$, how many compositions of n with k parts are there with skeleton $(m_j)_{j=1}^n$. The answer is (**exercise**)

$$\frac{k!}{m_1!m_2!\cdots m_n!}.$$

Also each composition having skeleton $(m_j)_{j=1}^n$ contributes

$$\frac{n!}{k!}\prod_{j=1}^n \frac{w_j^{m_j}}{(j!)^{m_j}}$$

in $S_{n,k}((_\bullet\mathbf{w}))$ by Lemma 3.1. Thus the coefficient of $w_1^{m_1}w_2^{m_2}\cdots w_n^{m_n}$ in $B_{n,k}(\mathbf{w}_\bullet)$ is

$$\frac{n!}{k!}\cdot\prod_{j=1}^n \frac{1}{(j!)^{m_j}}\cdot\frac{k!}{m_1!m_2!\cdots m_n!} = \frac{n!}{\prod_{j\geqslant 1}(j!)^{m_j}\cdot m_j!}.$$

∎

3.1. **Moments and Cumulants.** Given a random variable X and an integer $n \geqslant 1$, its n-th moment is defined as

$$\mu_n := \mathbb{E}(X^n)$$

when it exists. Let

$$m_X(t) = 1 + \sum_{n=1}^\infty \mu_n \frac{t^n}{n!} = \mathbb{E}(e^{tX})$$

be the moment generating function of X, if it exists.

Definition 3.4. *The n-th cumulant*

$$\kappa_n := \left[\frac{t^n}{n!}\right]\log m_X(t)$$

of X is defined as the coefficient of $\frac{t^n}{n!}$ in $\log m_X(t)$ for $n \geqslant 1$.

Example 3.5. *Let X ~$N(\mu,\sigma^2)$, normal distribution with mean μ and variance σ^2. The moment generating function of X is $m_X(t) = \exp(\mu t + \sigma^2 t^2/2)$ with $\log m_X(t) = \mu t + \sigma^2 t^2/2$. Thus the cumulants are $\kappa_1 = \mu, \kappa_2 = \sigma^2, \kappa_3 = \kappa_4 = \cdots = 0$.*

Example 3.6. *Let X ~Poisson(λ), Poisson distribution with mean λ. The moment generating function of X is $m_X(t) = \exp(\lambda(e^t - 1))$ with $\log m_X(t) = \lambda(e^t-1) = \lambda\sum_{n=1}^\infty \frac{t}{n!}$. Thus the cumulants are $\kappa_n = \lambda$ for $n \geqslant 1$.*

In general it is not easy to find the values of κ_n directly as we first have to calculate the moment generating function and then expand its logarithm in a power series. However, the following result shows how to find cumulants in terms of the moments.

Lemma 3.7. *Let $\mu_\bullet = (\mu_n)_{n\geqslant 1}$ be the moments of a random variable X. Then its n-th cumulant κ_n is given by*

$$\kappa_n = \sum_{k=1}^n (-1)^{k-1}(k-1)!B_{n,k}(\mu_\bullet).$$

Proof. Note that

$$\kappa_n := \left[\frac{t^n}{n!}\right]\log m_X(t)$$

where $m_X(t) = 1 + g(t)$ and $g(t)$ is the exponential generating function associated with the sequence $\mu_\bullet = (\mu_n)_{n\geqslant 1}$. Thus

$$\kappa_n := \left[\frac{t^n}{n!}\right]\log(1+g(t)) = \left[\frac{t^n}{n!}\right]\sum_{k=1}^\infty(-1)^{k-1}\frac{g(t)^k}{k} = \sum_{k=1}^\infty (-1)^{k-1}(k-1)!\left[\frac{t^n}{n!}\right]\frac{g(t)^k}{k!}.$$

By Lemma 3.2 the proof follows. ∎

4. Composite structures

Let $\mathbf{v}_\bullet := (v_1, v_2, \ldots)$ and $\mathbf{w}_\bullet := (w_1, w_2, \ldots)$ be two sequences of non-negative integers. Let V be some species of combinatorial structures, so for each finite set F_n on $n$objects there is some construction of a set $V(F_n)$ of V-structures on F_n, such that the number of V-structures on a set of n objects is $|V(F_n)| = v_n$. For instance, $V(F_n)$ might be $F_n \times F_n$, or permutations from F_n to F_n, or functions from F_n to F_n, or all graphs on F_n corresponding to the sequences $v_n = n^2$, or $n!$, or n^n, or $2^{n(n-1)/2}$ respectively.

Let W be another species of combinatorial structures, such that the number of W-structures on a set of j objects is w_j. Let $(V \circ W)(F_n)$ denote the composite structure on F_n defined as the set of all ways to partition F_n into blocks $\{A_1, \ldots, A_k\}$ for some $1 \leqslant k \leqslant n$, assign this collection of blocks a V-structure, and assign each block A_i a W-structure. Then for each set F_n with n elements, the number of such composite structures is evidently

$$|(V \circ W)(F_n)| = \sum_{k=1}^{n} v_k \sum_{\{A_1, A_2, \ldots, A_k\} \in \mathcal{P}^k_{[n]}} \prod_{i=1}^{k} w_{|A_i|} = \sum_{k=1}^{n} v_k B_{n,k}(\mathbf{w}_\bullet).$$

We define

$$B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet) := \sum_{k=1}^{n} v_k B_{n,k}(\mathbf{w}_\bullet).$$

Thus we have the following result.

Lemma 4.1. *Let*

$$v(x) := \sum_{n \geqslant 1} v_n \frac{z^n}{n!}, w(x) := \sum_{n \geqslant 1} w_n \frac{z^n}{n!}$$

be the exponential generating functions associated with the sequences $\mathbf{v}_\bullet := (v_1, v_2, \ldots)$ *and* $\mathbf{w}_\bullet := (w_1, w_2, \ldots)$*, respectively. Then we have*

$$\left[\frac{x^n}{n!}\right](v \circ w)(x) = B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet).$$

Proof. The proof follows from the fact that

$$(v \circ w)(x) = \sum_{k=1}^{\infty} v_k \frac{w(x)^k}{k!}$$

and Lemma 3.2. ∎

Example 4.2 (Partitions)**.** *Let V be the trivial structure, namely the set itself. Thus $v_k \equiv 1$. It is easy to see that $v(x) = e^x - 1$ and thus $v(v(x)) = e^{e^x - 1} - 1$. The composite structure $(V \circ V)([n])$ is a partition of $[n]$ with any number of parts. Moreover, $|(V \circ V)([n])|$, total number of partitions of $[n]$ is $\left[\frac{x^n}{n!}\right](e^{e^x - 1} - 1)$.*

Example 4.3 (Permutations)**.** *Let V be the trivial structure, namely the set itself and W be the cycle structure, namely given n objects V gives the set itself and W gives a cyclic permutation of the n objects. Thus $v_k \equiv 1$ and $w_n = (n-1)!$. It is easy to see that $v(x) = e^x - 1$ and $w(y) = \sum_{n=1}^{\infty} (n-1)! y^n / n! = \sum_{n=1}^{\infty} y^n / n = -\log(1-y)$. Thus $v(w(x)) = e^{-\log(1-x)} - 1 = \frac{x}{1-x} = \sum_{n=1}^{\infty} x^n$. In particular, we have $(V \circ W)([n]) = B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet) = n!$, which is the number of permutations on $[n]$. In fact, one can check that the composite structure $(V \circ W)$ is indeed the permutation structure.*

Example 4.4. *Let Vbe the permutation structure. Thus $v_k = k!$ and $v(x) = \sum_{k=1}^{\infty} x^k = x/(1-x)$. Moreover, $v(v(x)) = v(x)/(1 - v(x)) = \frac{x}{1-x} \frac{1-x}{1-2x} = \sum_{n=1}^{\infty} 2^{n-1} x^n$. Thus $|(V \circ V)([n])| = 2^{n-1} n!$. The same result can be obtained, by noting that the $(V \circ V)$ composite structure on $[n]$ is nothing but taking a permutation of $[n]$ in $n!$ ways and in between two consecutive elements either put a ‘,’ or no ‘,’ in 2^{n-1} ways.*

5. Random sums

Let $X, X_1, X_2, \ldots$ be independent and identically distributed nonnegative integer valued random variables with *probability generating function*

$$G_X(z) = \mathbb{E}(z^X) = \sum_{n=0}^{\infty} \mathbb{P}(X = n) z^n.$$

Let K be another non-negative integer valued random variable independent of $X_1, X_2, \ldots$ with *probability generating function* G_K. Let

$$S_K := X_1 + X_2 + \cdots + X_K.$$

By conditioning on K, the probability generating function of S_K is found to be the composition of G_K and G_X:

$$G_{S_K}(z) = G_K(G_X(z)).$$

Comparison of this formula with the compositional formula given in Lemma 4.1 for $B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$ in terms of the exponential generating functions $v(z) = \sum_{n\geqslant 1} v_n z^n/n!$ and $w(\xi) = \sum_{n\geqslant 1} w_n \xi^n/n!$ suggests the following construction (It is convenient here to allow v_0 to be non-zero, which makes no difference in Lemma 4.1).

Let $\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}$ be a probability distribution which makes X_i independent and identically distributed with the power series distribution

$$\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(X = n) = \frac{w_n \xi^n}{n! w(\xi)} \text{ for } n = 1, 2, \ldots$$

so that

$$G_X(z) = \frac{w(z\xi)}{w(\xi)}$$

and K independent of X_i's with the power series distribution

$$\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(K = k) = \frac{v_k w(\xi)^k}{k! v(w(\xi))} \text{ for } k = 1, 2, \ldots$$

so that

$$G_K(y) = \frac{v(yw(\xi))}{v(w(\xi))}.$$

Let $S_K := X_1 + X_2 + \cdots + X_K$. Then we have

$$\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(S_K = n) = [z^n](G_K \circ G_X)(z) = [z^n]\frac{v(w(z\xi))}{v(w(\xi))} = \frac{\xi^n}{n! v(w(\xi))} B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$$

or

$$B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet) = \frac{n! v(w(\xi))}{\xi^n} \mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(S_K = n).$$

This gives a probabilistic interpretation of $B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$. In general one can choose ξ so that $\mathbb{E}(S_K) = \mathbb{E}(K)\mathbb{E}(X) = n$ and use local limit theorem for $\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(S_K = n)$ to obtain good estimate for $B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$.

6. Gibbs Partition

Suppose that $(V \circ W)([n])$ is the set of all $(V \circ W)$ composite structures built over the set $[n]$, for some species of combinatorial structures V and W. Let a composite structure be picked uniformly at random from $(V \circ W)([n])$, and let Π_n denote the random partition of $[n]$ generated by blocks of this random composite structure. Recall that v_j and w_j denote the number of V- and W- structures respectively on a set of j objects. Then for each particular partition $\{A_1, A_2, \ldots, A_k\}$ of $[n]$ it is clear that

$$\mathbb{P}(\Pi_n = \{A_1, A_2, \ldots, A_k\}) = p(|A_1|, |A_2|, \ldots, |A_k|; \mathbf{v}_\bullet, \mathbf{w}_\bullet) \tag{2}$$

where for each composition $(n_1, n_2, \ldots, n_k)$ of n we have

$$p(n_1, n_2, \ldots, n_k; \mathbf{v}_\bullet, \mathbf{w}_\bullet) = \frac{v_k \prod_{i=1}^k w_{n_i}}{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)}$$

with the normalizing constant

$$B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet) = \sum_{k=1}^{n} v_k B_{n,k}(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$$

assumed to be strictly positive. More generally, given two non-negative sequences $\mathbf{v}_\bullet := (v_1, v_2, \ldots)$ and $\mathbf{w}_\bullet := (w_1, w_2, \ldots)$, call Π_n a $\text{Gibbs}_{[n]}(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$ partition if the distribution of Π_n on $\mathcal{P}_{[n]}$ is given by (2).

Note that we have the following redundancy in the parameterization of $\text{Gibbs}_{[n]}(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$ partitions: for arbitrary positive constants a, b and c,

$$\text{Gibbs}_{[n]}(\tilde{\mathbf{v}}_\bullet, \tilde{\mathbf{w}}_\bullet) = \text{Gibbs}_{[n]}(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$$

where $\tilde{\mathbf{v}}_\bullet = (ab^{-1}v_1, ab^{-2}v_2, ab^{-3}v_3, \ldots)$ and $\tilde{\mathbf{w}}_\bullet = (bcw_1, bc^2w_2, bc^3w_3, \ldots)$.

6.1. **The block sizes in exchangeable order.** The following theorem provides a fundamental representation of Gibbs partitions.

Theorem 6.1 (Kolchin's representation of Gibbs partitions). *Let $(N^{ex}_{n,1}, N^{ex}_{n,2}, \ldots, N^{ex}_{n,|\Pi_n|})$ be the random composition of n defined by putting the block sizes of a Gibbs$_{[n]}(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$ partition Π_n in an exchangeable random order, meaning that given k blocks, the order of the blocks is randomized by a uniform random permutation of $[k]$. Then*

$$(N^{ex}_{n,1}, N^{ex}_{n,2}, \ldots, N^{ex}_{n,|\Pi_n|}) \stackrel{\mathrm{d}}{=} (X_1, X_2, \ldots, X_k) \text{ under } \mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet} \text{ given } X_1 + X_2 + \cdots + X_K = n$$

where $\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}$ governs independent and identically distributed random variables $X_1, X_2, \ldots$ with $\mathbb{E}(z^{X_1}) = w(z\xi)/w(\xi)$ and K is independent of these variables with $\mathbb{E}(y^K) = v(yw(\xi))/v(w(\xi))$.

Proof. Note that we have for any composition $(n_1, n_2, \ldots, n_k)$ of n,

$$\begin{aligned}\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(X_1 = n_1, X_2 = n_2, \ldots, X_k = n_k, K = k) &= \mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(K = k) \prod_{i=1}^{k} \mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(X = n_i) \\ &= \frac{v_k w(\xi)^k}{k! v(w(\xi))} \prod_{i=1}^{k} \frac{w_{n_i} \xi^{n_i}}{n_i! w(\xi)} = \frac{v_k \xi^n}{k! v(w(\xi))} \prod_{i=1}^{k} \frac{w_{n_i}}{n_i!}\end{aligned}$$

and

$$\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(S_K = n) = \frac{\xi^n}{n! v(w(\xi))} B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet).$$

Thus

$$\begin{aligned}&\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}((X_1, X_2, \ldots, X_K) = (n_1, n_2, \ldots, n_k) \mid S_K = n) \\ &= \frac{\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}((X_1, X_2, \ldots, X_K) = (n_1, n_2, \ldots, n_k))}{\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}(S_K = n)} = \frac{1}{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)} \cdot v_k \frac{n!}{k!} \prod_{i=1}^{k} \frac{w_{n_i}}{n_i!}.\end{aligned}$$

Now Lemma 3.1 can be interpreted probabilistically to show that

$$\mathbb{P}((N^{\mathrm{ex}}_{n,1}, N^{\mathrm{ex}}_{n,2}, \ldots, N^{\mathrm{ex}}_{n,|\Pi_n|}) = (n_1, n_2, \ldots, n_k)) = \frac{1}{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)} \cdot v_k \frac{n!}{k!} \prod_{i=1}^{k} \frac{w_{n_i}}{n_i!}$$

for all compositions $(n_1, n_2, \ldots, n_k)$ of n. Thus the two distributions are same. ■

Note that for fixed $\mathbf{v}_\bullet$ and $\mathbf{w}_\bullet$, the $\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}$ distribution of the random integer composition $(X_1, X_2, \ldots, X_K)$ depends on the parameter ξ, but the $\mathbb{P}_{\xi,\mathbf{v}_\bullet,\mathbf{w}_\bullet}$ conditional distribution of $(X_1, X_2, \ldots, X_K)$ given $S_K = n$ does not. In statistical terms, with $\mathbf{v}_\bullet$ and $\mathbf{w}_\bullet$ regarded as fixed and known, the sum S_K is a sufficient statistic for ξ. Note also that for any fixed n, the distribution of Π_n depends only on the weights v_j and w_j for $j \leqslant n$, so the condition $v(w(\xi)) < \infty$ can always be arranged by setting $v_j = w_j = 0$ for $j > n$.

Example 6.2 (Random partitions). *Take $v_k = w_k \equiv 1$, so that $v(x) = w(x) = e^x - 1$. Let X have distribution*

$$\mathbb{P}(X = n) = \frac{\xi^n}{n!(e^\xi - 1)}, n \geqslant 1,$$

K have distribution

$$\mathbb{P}(K = k) = \frac{(e^\xi - 1)^k}{k!(e^{e^\xi - 1} - 1)}, k \geqslant 1$$

and $X_1, X_2, \ldots$ be i.i.d. copies of X independent of K. Then the block sizes of a uniform random partition of $[n]$ (ordered in an exchangeable way using random ordering) has distribution

$$(X_1, X_2, \ldots, X_K) \text{ given } X_1 + X_2 + \cdots + X_K = n.$$

Note that $\mathbb{E}(X) = \sum_{n=1}^\infty n \cdot \frac{\xi^n}{n!(e^\xi - 1)} = \frac{\xi e^\xi}{e^\xi - 1} = \frac{\xi}{1 - e^{-\xi}}$, $\mathbb{E}(K) = \frac{e^\xi - 1}{1 - e^{1 - e^\xi}}$ and $\mathbb{E}(S_K) = \mathbb{E}(X)\mathbb{E}(K) = \frac{\xi e^\xi}{1 - e^{1 - e^\xi}}$. Thus if we choose $\xi = \log n - \log\log n$, we have $\mathbb{E}(S_K) \approx n$, $\mathbb{E}(K) \approx n/\log n$ and $\mathbb{E}(X) \approx \log n - \log\log n$. This suggests that in a uniform random permutation we have typically $n/\log n$ many blocks and the average block size is $\log n - \log\log n$.

ST329: Exercise sheet 1

(1) Give a combinatorial proof of the identity

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}.$$

Hint: Consider n red balls and m white balls and count number of ways to choose k balls from them without repetition.

(2) Give a combinatorial proof of the identity

$$S(n,k) = kS(n-1,k) + S(n-1,k-1)$$

where $S(n,k)$ is the Stirling number of second kind.
Hint: See the notes.

(3) Give a combinatorial proof of the identity

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k$$

where $S(n,k)$ is the Stirling number of second kind and $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ is the falling factorial.
Hint: Use the fact that for an integer x, x^n is the number of ways to arrange x distinguishable objects into n ordered places with repetition of objects allowed.

(4) Let $g_1(x) = e^x - 1$. Let $g_k(x), k \geqslant 1$ be differentiable functions such that $g_k(0) = 0$ and the following holds:

$$g_k'(x) = kg_k(x) + g_{k-1}(x)$$

for all x, k. Show by induction that

$$g_k(x) = \frac{1}{k!}(e^x - 1)^k$$

for all k.
Hint: Clearly g_1 is of the given form. If $g_{k-1}(x)$ is given by the above formula, show that $g_k(x)$ is also given by the above formula. Use the fact that $(e^{-kx}g_k(x))' = e^{-kx}(g_k'(x) - kg_k(x))$.

(5) Show that the number of weak composition of n into k parts is $\binom{n+k-1}{k-1}$ and number of composition of n into k parts is $\binom{n-1}{k-1}$. Use this result to give a combinatorial proof of the identity

$$\sum_{i=1}^{k} \binom{n-1}{i-1}\binom{k}{i} = \binom{n+k-1}{k-1}.$$

Hint: Consider the places with 0 in the weak compositions.

(6) Let F_n be the number of compositions of the integer n such that all the terms are odd.
(a) Show that $F_1 = 1, F_2 = 1$ and F_n satisfies $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 3$.
(b) Let $f(x) = \sum_{n\geqslant 1} F_n x^n$ be the generating function of the sequence $F_n, n \geqslant 1$. Show that

$$f(x) = x + xf(x) + x^2 f(x)$$

using the relation in (a).
(c) Show that $f(x) = \frac{x}{1-x-x^2}$. The sequence $(F_n, n \geqslant 1)$ is known as Fibonacci sequence.

ST329: Exercise sheet 2

(1) Let $(p_1, p_2, \ldots)$ be a sequence of numbers such that $\sum_{i \geqslant 1} p_i = k$ and $\sum_{i \geqslant 1} i p_i = n$. Show that the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k blocks such that there are p_i many blocks of size i for $i \geqslant 1$, is

$$\frac{n!}{\prod_{j \geqslant 1} (j!)^{p_j} p_j!}.$$

(2) Prove that

$$B_{n,2}(\mathbf{w}_\bullet) = \begin{cases} \sum_{i=1}^{(n-1)/2} \binom{n}{i} w_i w_{n-i} & \text{if } n \text{ is odd} \\ \sum_{i=1}^{n/2} \binom{n}{i} w_i w_{n-i} - \frac{1}{2} \binom{n}{n/2} w_{n/2}^2 & \text{if } n \text{ is even.} \end{cases}$$

Hint: Done in the class.

(3) Show that the number of graphs on $[n]$ is $2^{\binom{n}{2}}$.
Hint: Done in the class.

(4) Let W be the connected graph structure and let w_n be the number of connected graphs on $[n]$. Let V be the trivial structure with $v_n \equiv 1$. Show that $(V \circ W)([n])$ is the graph structure on $[n]$. Moreover, using the theory about composite structures show that

$$w_n = \left[\frac{x^n}{n!}\right] \log\left(1 + \sum_{k=1}^{\infty} 2^{\binom{k}{2}} \frac{x^k}{k!}\right).$$

Hint: Done in the class.

(5) Let V be the trivial structure and let W be the cycle structure, so that $(V \circ W)$ composite structure is the permutation structure. Using this connection find the distribution of the cycle lengths in a uniform random permutation. Give a heuristic behind the statement that there are roughly $\log n$ many cycles in a uniform random permutation, similar to Example 6.2.

(6) Let $\mathbf{v}_\bullet = (1, 1, 1, \ldots)$ and $\mathbf{w}_\bullet = (w_1, w_2, \ldots)$. Let Π_n be a $\text{Gibbs}_{[n]}(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$ partition. Let $|\Pi_n|_j$ be the number of blocks of Π_n of size j. Show that,

$$(|\Pi_n|_j)_{j=1}^n \stackrel{\mathrm{d}}{=} \left((M_j)_{j=1}^n \,\middle|\, \sum_{j=1}^n j M_j = n \right)$$

where the M_j are independent Poisson variables with parameters $w_j \xi^j / j!$ for arbitrary $\xi > 0$.
Hint: Given a sequence of integers $m_j, j = 1, 2, \ldots, n$ with $\sum_{j \geqslant 1} j m_j = n$, show that $\mathbb{P}(|\Pi_n|_j = m_j, j = 1, 2, \ldots, n)$ is same as $\mathbb{P}(M_j = m_j, j = 1, 2, \ldots, n | \sum_{j=1}^n j M_j = n)$. For the first probability use Lemma 3.3 and for the second one use Bayes' rule for conditional probability and independence of M_j's.

(1) **Combinatorial Stochastic Process.**

(a) Show that the number of weak composition of n into k parts is $\binom{n+k-1}{k-1}$ and number of composition of n into k parts is $\binom{n-1}{k-1}$.

[4 marks]

(b) Use the result in (a) to give a combinatorial proof of the identity

$$\sum_{i=1}^{k} \binom{n-1}{i-1}\binom{k}{i} = \binom{n+k-1}{k-1}.$$

[2 marks]

(c) Let $(p_1, p_2, \ldots)$ be a sequence of numbers such that $\sum_{i \geqslant 1} p_i = k$ and $\sum_{i \geqslant 1} i p_i = n$. Show that the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k blocks such that there are p_i many blocks of size i for $i \geqslant 1$, is

$$\frac{n!}{\prod_{j \geqslant 1} (j!)^{p_j} p_j!}.$$

[5 marks]

(d) Let $\mathbf{v}_\bullet = (v_1, v_2, \ldots)$ and $\mathbf{w}_\bullet = (w_1, w_2, \ldots)$. Let Π_n be a $\text{Gibbs}_{[n]}(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$ random partition and $\{A_1, A_2, \ldots, A_k\}$ be a fixed partition of $[n]$. Find the probability that $\Pi_n = \{A_1, A_2, \ldots, A_k\}$.

[1 mark]

(e) Assume that $\mathbf{v}_\bullet = (1, 1, 1, \ldots)$ and $\mathbf{w}_\bullet = (w_1, w_2, \ldots)$ is arbitrary. Let Π_n be a $\text{Gibbs}_{[n]}(\mathbf{v}_\bullet, \mathbf{w}_\bullet)$ partition. Let $|\Pi_n|_j$ be the number of blocks of Π_n of size j. Show that,

$$(|\Pi_n|_j)_{j=1}^{n} \stackrel{\mathrm{d}}{=} \left((M_j)_{j=1}^{n} \,\middle|\, \sum_{j=1}^{n} j M_j = n \right)$$

where the M_j are independent Poisson variables with parameters $w_j \xi^j / j!$ for arbitrary $\xi > 0$.

[8 marks]

(1) We consider $(n+m)$ many balls out of which n are red balls and m are white balls. Number of ways to choose k balls (the order does not matter) from them without repetition is $\binom{n+m}{k}$. We can also count this number in the following way: first fix i from $\{0,1,\ldots,k\}$, choose i red balls and $(k-i)$ white balls. Number of ways to choose i red balls from n balls is $\binom{n}{i}$ and number of ways to choose $(k-i)$ white balls from m balls is $\binom{m}{k-i}$. By multiplication principle the total number for the second way of counting is $\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}$. Thus we have

$$\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}=\binom{n+m}{k}.$$

(2) See Lemma 2.2.

(3) Fix an integer x. The number of ways to arrange x distinguishable objects into n ordered places with repetition of objects allowed is x^n. We can also count the number in the following way: We name the places by $1,2,\ldots,n$. We fix an integer k from $[n]$; choose k ordered objects from x many distinguishable objects without repetition and a partition of $[n]$ into k blocks. Order the blocks according to the smallest number in the blocks (for example for the partition $\{14,5,23\}$ the first block is 13, second block is 23 and third block is 5). Now put the first chosen object in the places given in the first block, second chosen object in the places given in the second block and so on. Number of ways to choose k ordered objects from x many distinguishable objects without repetition is $(x)_k$ and number of partitions of $[n]$ into k blocks is $S(n,k)$. Thus we have $x^n=\sum_{k=0}^{n}S(n,k)(x)_k$.

(4) We consider n indistinguishable balls and $(k-1)$ indistinguishable bars. Number of ways to arrange them in order is $\binom{n+k-1}{k-1}$. Each arrangement gives a weak composition of n in the following way. Let the number of balls before the first bar be x_1, number of balls between the first and second bar be x_2 and so on, number of balls after the $(k-1)$-th bar be x_k. Clearly $(x_1,x_2,\ldots,x_k)$ gives a weak composition of n into k parts and any such weak composition arises in this way. Thus the number of weak compositions of n into k parts is $\binom{n+k-1}{k-1}$. To count the number of composition of n into k parts, we notice that we have to choose $(k-1)$ gaps out of the $(n-1)$ gaps between the n balls.
To prove the given identity, we count number of weak compositions of of n into k parts in a different way. If we ignore all the zeros from the weak composition, we will get a composition of n but likely to be with less than k parts. So we can first choose i (number of parts if we ignore all the zeroes in the weak composition) from $[k]$, a composition of n into i blocks and $(k-i)$ places out of k places where we will put the zeroes in the weak composition. For example, for $n=5,k=3$, choosing $i=2$, the composition $(3,2)$ of n into i parts and $k-i=1$ places $\{2\}$ gives the weak composition $(3,0,2)$. Number of compositions of n into i blocks is $\binom{n-1}{i-1}$ and number ways to choose $(k-i)$ places out of k places is $\binom{k}{k-i}=\binom{k}{i}$. By multiplication principle and summing, we have the number of weak compositions of n into k parts is

$$\sum_{i=1}^{k}\binom{n-1}{i-1}\binom{k}{i}.$$

(5) (a) Let us call a composition with all the terms being odd an "odd composition". Consider an odd composition of the integer n. If the composition contains 1, then after removing 1 we obtain an odd composition of $(n-1)$. If the composition does not contain 1, then all the terms are bigger than or equal to 3 and thus after subtracting 2 from the smallest term in the composition we obtain an odd composition of $(n-2)$. Similarly, given an odd composition of $(n-1)$, by attaching 1 to the composition we can construct an odd composition of n and given an odd composition of $(n-2)$, by adding 2 to the smallest term of the composition we can construct an odd composition of n. Thus we have $F_n=F_{n-1}+F_{n-2}$. Obviously $F_1=F_2=1$ as $1=1$ and $2=1+1$ each has only one odd composition. (b) we have $f(x)=\sum_{n\geqslant 1}F_nx^n=F_1x+F_2x^2+\sum_{n\geqslant 3}(F_{n-1}+F_{n-2})x^n=x+x^2+x\sum_{n\geqslant 3}F_{n-1}x^{n-1}+x^2\sum_{n\geqslant 3}F_{n-2}x^{n-2}=x+xf(x)+x^2f(x)$. (c) trivial.

Solutions to Exercise sheet 2

(1) Consider the sequence $(x_1, x_2, \ldots, x_k)$ where $x_1 = \cdots = x_{p_1} = 1$, $x_{p_1+1} = \cdots = x_{p_1+p_2} = 2$, $x_{p_1+p_2+1} = \cdots = x_{p_1+p_2+p_3} = 3$ and so on. Clearly $\sum_{i=1}^k x_i = \sum_{j\geqslant 1} jp_j = n$. By multiplication principle, number of ways to choose an ordered sequence of blocks so that the i-th block is of size x_i for $i = 1, 2, \ldots, k$, is

$$\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_1-x_2}{x_3}\cdots\binom{n-x_1-x_2-\cdots-x_{k-1}}{x_k} = \frac{n!}{\prod_{i\geqslant 1} x_i!} = \frac{n!}{\prod_j j!^{p_j}}.$$

However, in the partition of $[n]$ the ordering of the blocks does not matter, hence the p_j many blocks of size j, for $j \geqslant 1$ must be unordered to get unique partitions. Thus the number of ways to partition $[n]$ into k blocks such that the there are p_j many blocks of size j for $j \geqslant 1$, is

$$\frac{n!}{\prod_{j\geqslant 1} j!^{p_j} p_j!}.$$

(2) Done in the class.

(3) Done in the class.

(4) Done in the class.

(5) Similar to Example 6.2.

(6) First of all note that it is enough to show that $\mathbb{P}(|\Pi_n|_j = m_j, j = 1, 2, \ldots, n) = C \cdot \mathbb{P}(M_j = m_j, j = 1, 2, \ldots, n | \sum_{j=1}^n jM_j = n)$ for all sequence of integers $m_j, j = 1, 2, \ldots, n$ for some constant C that does not depend on $m_j, j = 1, 2, \ldots, n$. Since total probability is 1, C must automatically be equal to 1.

Note that both the probabilities are zero unless we have $\sum_{j\geqslant 1} jm_j = n$. Now take a sequence of integers $m_j, j = 1, 2, \ldots, n$ satisfying $\sum_{j=1}^n jm_j = n$. By definition of Gibbs partition, given any partition $\{A_1, A_2, \ldots, A_k\}$ of $[n]$ such that m_j of the A_i's have size j for $j = 1, 2, \ldots, n$, we have

$$\mathbb{P}(\Pi_n = \{A_1, A_2, \ldots, A_k\}) = \frac{v_k \prod_{i=1}^k w_{|A_i|}}{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)} = \frac{\prod_{j=1}^n w_j^{m_j}}{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)}.$$

By Question (1), the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k blocks such that there are m_j many blocks of size j for $j = 1, 2, \ldots, n$, is

$$\frac{n!}{\prod_{j=1}^n (j!)^{m_j} m_j!}.$$

Thus we have

$$\mathbb{P}(|\Pi_n|_j = m_j, j = 1, 2, \ldots, n) = \frac{n!}{\prod_{j=1}^n (j!)^{m_j} m_j!} \cdot \frac{\prod_{j=1}^n w_j^{m_j}}{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)} = \frac{n!}{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)} \cdot \prod_{j=1}^n \frac{w_j^{m_j}}{(j!)^{m_j} m_j!}.$$

Now, by Bayes' rule we have

$$\begin{aligned}\mathbb{P}(M_j = m_j, j = 1, 2, \ldots, n | \sum_{j=1}^n jM_j = n) &= \frac{\mathbb{P}(M_j = m_j, j = 1, 2, \ldots, n, \sum_{j=1}^n jM_j = n)}{\mathbb{P}(\sum_{j=1}^n jM_j = n)} \\ &= \frac{\mathbb{P}(M_j = m_j, j = 1, 2, \ldots, n)}{\mathbb{P}(\sum_{j=1}^n jM_j = n)}.\end{aligned}$$

Moreover, by independence of M_j's and definition of Poisson distribution we have

$$\begin{aligned}\mathbb{P}(M_j = m_j, j = 1, 2, \ldots, n | \sum_{j=1}^{n} jM_j = n) &= \frac{\prod_{j=1}^{n} \mathbb{P}(M_j = m_j)}{\mathbb{P}(\sum_{j=1}^{n} jM_j = n)} \\ &= \frac{1}{\mathbb{P}(\sum_{j=1}^{n} jM_j = n)} \prod_{j=1}^{n} e^{-w_j \xi^j / j!} \frac{(w_j \xi^j / j!)^{m_j}}{m_j!} \\ &= \frac{\xi^n e^{-\sum_{j=1}^{n} w_j \xi^j / j!}}{\mathbb{P}(\sum_{j=1}^{n} jM_j = n)} \prod_{j=1}^{n} \frac{w_j^{m_j}}{(j!)^{m_j} m_j!}\end{aligned}$$

In particular, we have

$$\mathbb{P}(|\Pi_n|_j = m_j, j = 1, 2, \ldots, n) = C \cdot \mathbb{P}(M_j = m_j, j = 1, 2, \ldots, n | \sum_{j=1}^{n} jM_j = n)$$

for all sequence of integers $m_j, j = 1, 2, \ldots, n$ with $C = \frac{n!}{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)} \cdot \frac{\mathbb{P}(\sum_{j=1}^{n} jM_j = n)}{\xi^n e^{-\sum_{j=1}^{n} w_j \xi^j / j!}}$. This completes the proof.

As a byproduct we obtain that,

$$\mathbb{P}(\sum_{j=1}^{n} jM_j = n) = \frac{B_n(\mathbf{v}_\bullet, \mathbf{w}_\bullet)}{n!} \xi^n e^{-\sum_{j=1}^{n} w_j \xi^j / j!}.$$